Journal of Guidance, Control, and Dynamics
Vol. 32, No. 5, September–October 2009

Optimal Satellite Transfers Using Relative Motion Dynamics

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DOI: 10.2514/1.42847

A first-order approximation of optimal two-burn transfers between satellite orbits within a formation is presented and discussed. The orbital transfer problem is based on the well-known fact that, to first order, such orbits are elliptic, with centers at the target or offset in the in-track direction. The methodology allows for a target in a (generally) elliptic orbit in its motion about the center of force. The solution obtained makes use of a transformation of the problem to the reference frame of an auxiliary target in circular orbit in the inertial frame. The total required $|\Delta v|$ is expressed in terms of the identifying parameters for the initial and final orbits in the formation, corresponding to their respective sizes as measured by their minor axes, location of their centers, and their orientation. This expression allows for numerical parametric studies in the case where the initial and final orbits are in the same or parallel planes. It is shown that the optimal total $|\Delta v|$ depends on the relative sizes of the orbits and the distance between their centers. When the initial and final orbits are concentric, four optimal two-burn transfers, all of the same total $|\Delta v|$, are identified. It is further shown that the optimal $|\Delta v|$ changes with the distance between the centers of the orbits and attains a minimum when this distance is about 2.2 times the difference between the minor axes of the orbits.

Introduction

THE use of spacecraft formations in various applications, ranging from the study of the ionosphere [1] to telescopic observations of Earth, making use of aperture synthesis [2], is a promising technology that is bound to be of increasing importance. A survey of satellite formation flying is given in [3].

A common way of defining and describing a satellite formation is the use of a localized reference frame that is attached to one member satellite of the formation (designated the target or leader) and to consider the motions of the other member satellites (designated the chasers or followers) in this reference frame. This approach allows for the derivation of the equations of motion of the chasers in the target's reference frame. The equations of motion are useful for the development of control schemes for the desired configurations and for the implementation of formation-related maneuvers, such as reconfiguration, relative stationkeeping, etc.

The first-order equations of motion for a chaser's relative motion, with respect to a target on a circular orbit about the center of force, are the well-known linearized Hill's equations, also known as the Clohessy–Wiltshire equations [4]. Many contributions to the study of the dynamics of satellite formations have been based on these equations: to name a few, formation flying designs and their time evolution are studied in [5]; the use of formation flying satellites for aperture synthesis is studied in [2]; in [6], a study of correctional maneuvers to mitigate the effects of perturbations such as the J_2 effect, atmospheric drag, and solar radiation pressure is presented; and control design for the Orion mission is studied in [7]. Furthermore, the Clohessy–Wiltshire equations are extended to account for J_2 perturbations in [8]. In [9], a third-order analytical solution to the Clohessy–Wiltshire equations, based on perturbation methods, is given.

The Clohessy–Wiltshire equations have also been used for the purpose of optimal control of satellite formations. For example, in [10], mixed-integer programming is used for trajectory planning with constraint avoidance, and more recently, in [11], studies of optimal multi-objective impulsive rendezvous are reported. In addition, the optimal relocation of satellites using continuous thrust is studied in [12].

The Clohessy–Wiltshire equations, though based on the assumption that the target is in a circular orbit, may sometimes be used as an approximate model for relative motion with respect to a target in a low eccentricity orbit. However, a more accurate treatment of this problem can be done by explicitly taking into account the eccentricity of the target's orbit. Thus, the equations of motion for the case where the target is on an elliptic orbit are used in, for example, [13–15] and the corresponding motions are studied. In [16], these equations serve as a basis for optimal formation control. As is noted in [16], the decrease of modeling errors that can be achieved by accounting for the eccentricity of the target's orbit leads to a reduction of fuel costs.

Although equations of relative motion are important for the implementation of control algorithms, one should keep in mind that the relative motion itself is readily available as the difference between the chaser's and target's motions with respect to the center of force (i.e., by subtracting the position vector of the target from that of the chaser), both of which are exactly known as solutions of the two-body problem. This approach is taken in [17], in which initial conditions for the relative motion that are accurate to second order in the amplitude of the relative motion are given. Such initial conditions considerably decrease the rate of drift of relative orbits.

When orbits within a formation are determined as the difference between the chaser's and target's orbits, one can express the size (minor and major axes) of the orbits in terms of the eccentricities of the target's and chaser's inertial orbits. Also, the condition for the periodicity of the relative motion (i.e., the absence of drift away from the target) can be expressed in terms of a condition on the semimajor axes of the underlying orbits (see, e.g., [17]).

The problem studied in this paper is motivated by the general practical importance of transferring a satellite between two given orbits that correspond to different relative motions with respect to the target. For example, such a transfer may be made to change the size of a synthetic aperture telescope or to make stationkeeping corrections to the orbits of a formation that have drifted away from a target. Also, the launching of secondary satellites from a target into orbit about the latter, for example, a nanosatellite launched from a spacecraft for in-orbit inspection of that spacecraft, is a special case of the same problem (i.e., with zero initial amplitude and offset).

The specific aim of this paper is to determine optimal two-burn transfers of a chaser between two orbits in a formation with respect to a target. For this purpose, it is often convenient to think directly in

Received 19 December 2008; revision received 14 May 2009; accepted for publication 15 May 2009.

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terms of the orbits about the target, as opposed to the underlying satellite orbits in the inertial frame. For simplicity of expression, the orbits in the reference frame of the target will be referred to as *relative orbits* in what follows. These periodic relative orbits (which are always elliptic with center at the target) are parametrized in terms of their size (semiminor axis), position of their center with respect to the target (center offset), relative inclination with respect to the local-vertical-local-horizontal (LVLH) coordinate system, as well as relative right ascension. The resulting transfers are expressed (to first order) in terms of the defining parameters of the initial and final orbits (minor axis, center location, relative right ascension, and relative inclination). Although such optimal transfer orbits can be determined accurately through, for example, numerical calculations using the relative equations of motion, the first-order analytical and numerical results in this paper are useful for obtaining preliminary estimates of the transfers. The problem is therefore analogous to that of determining optimal transfers, for example, between two circular orbits, in the two-body problem; whereas numerical methods are important in determining and implementing actual transfers, Hohmann transfers give preliminary analytical estimates of these.

In the optimal transfer problem in this paper, it is assumed that the target is generally in an elliptic orbit about the center of force, in the inertial frame. In the solution of the problem, however, it is shown that a transformation can be done to the reference frame of an auxiliary target in circular orbit. This result is important in that it makes available the analytical solutions of Hill's equations for the case of a target in circular orbit, for use in a problem with a target in an elliptic orbit. Using this approach, it is shown that, in the most general case, the total required $|\Delta\mathbf{v}|$ depends on two quantities: the difference between the eccentricities of the orbits in the inertial frame that correspond to the initial and final relative orbits, and the distance between the centers of the initial and final relative orbits. A concise expression of $|\Delta\mathbf{v}|$ is derived, in terms of the aforementioned quantities, that allows for the parametric study of $|\Delta\mathbf{v}|$ with respect to these quantities. In particular, for example, it is shown that $|\Delta\mathbf{v}|$ has a local minimum with respect to the difference in center offsets, and an example is given on how this may be of use in practice.

As has been mentioned earlier, a relative orbit in a target's reference frame corresponds to an underlying inertial orbit. Therefore, the optimal transfer between two relative orbits in a target's reference frame can be expressed as a transfer between the two underlying inertial orbits. In that way, the problem being studied in this paper is related to that of a transfer between two orbits about a center of force, which is a central problem of astrodynamics that has been studied by many authors (see, for example, [18–20]). In particular, the problem of optimal rendezvous, that is, where the time of perigee passage is specified on the initial and final orbits, is studied in, for example, [21–23]. However, all of these works assume a fixed time of rendezvous, that is, the time of transfer is fixed a priori. By contrast, in the current problem, the optimal time of transfer follows as a result of the optimization.

Periodic Relative Orbits

It is well known that small periodic relative satellite orbits about a given target are elliptic with a major axis equal to twice the minor axis. These orbits are best described through the use of a LVLH coordinate system with an origin at the target satellite. The x axis of this coordinate system points directly away from the center of force, whereas the y axis is perpendicular to the x axis, lies in the orbital plane of the target, and points in the direction of an increasing true anomaly. Lastly, the z axis completes the right-handed system.

The properties of relative orbits may also be described in terms of parameters of the underlying inertial orbits about the center of force. For example, it may be noted that periodic relative orbits appear when the semimajor axis of the chaser's inertial orbit a_C is equal to that of the target a_T, as otherwise the periods of the orbits will be different and drifting will result. In what follows, it will therefore be assumed that

$$a_T = a_C = a \tag{1}$$

Also, it is shown in [17] that if the eccentricities of the target's and chaser's orbits are e_T and e_C, respectively, then the minor axis of the elliptic relative orbit is $|a\delta|$ with

$$\delta = \sqrt{e_C^2 + e_T^2 - 2e_C e_T \cos E_{C0}} \tag{2}$$

and where E_{C0} is the eccentric anomaly of the chaser at the time when the eccentric anomaly of the target is zero. Further, the center of a relative orbit is generally offset in the y direction from the origin of the coordinate system (the target) by a distance that will be denoted here by y_0 (see Fig. 1). An offset for the center in the x direction is not possible for an orbit that does not drift with respect to the target (see [17]). Lastly, the relative orbit may be inclined with respect to the fundamental plane of the LVLH coordinate system by an angle that will be denoted here by γ. This angle is also the angle between the z axis and normal to the relative orbit's plane.

Now, defining

$$n = \sqrt{\frac{\mu}{a^3}} \tag{3}$$

the relative orbits may be expressed to first order in δ, y_0, and γ as (see [17] for details)

$$x = -\delta a \cos\phi \tag{4}$$

$$y = y_0 + 2\delta a \sin\phi \tag{5}$$

$$z = a\gamma \sin(\phi - \phi_{z0}) \tag{6}$$

where $\phi = nt + \phi_0$ is the angle to the position of the satellite measured positive clockwise from the negative x direction (see Fig. 1). Here, ϕ_0 has been defined as the value of ϕ at time $t = 0$. It is shown in [17] that ϕ can be written in terms of parameters for the underlying inertial orbit through

$$\phi_0 = \arctan\frac{e_C \sin E_{C0}}{e_T - e_C \cos E_{C0}} \tag{7}$$

The angle ϕ_{z0} is the value of ϕ at which the orbit will be crossing the x–y plane such that the z coordinates of points will change from negative to positive. It will be referred to as the relative right ascension. This is indicated in Fig. 1 through the use of a dashed line for the parts of the orbit below the x–y plane (i.e., $z < 0$).

In what follows, it is convenient to use canonical units (see, e.g., [24]) where the reference orbit is taken to be a circular orbit of semimajor axis a. Thus, a distance of 1 DU will correspond to the semimajor axis a. The gravitational parameter will have the value of $\mu = 1(\mathrm{DU}^3/\mathrm{TU}^2)$ and consequently $n = 1/\mathrm{TU}$. It follows then that Eqs. (4–6) can be written

$$x = -\delta \cos\phi \tag{8}$$

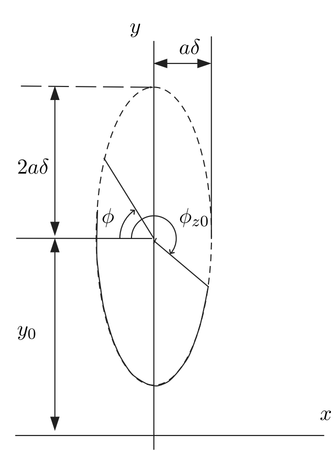

Fig. 1 Projection of a generic relative orbit on the *x*–*y* plane: dashed line indicates parts of the orbit with negative values of the *z* coordinate.

$$y = y_0 + 2\delta \sin\phi \tag{9}$$

$$z = \gamma \sin(\phi - \phi_{z0}) \tag{10}$$

It is worth noting that the relative orbit described previously corresponds to an inertial orbit of a chaser that is close to the inertial orbit of the target. Thus, with respect to another target close to the first one, and in an elliptic orbit of eccentricity, say e'_T, the same inertial orbit of the chaser would give rise to a relative orbit of minor axis

$$\delta = \sqrt{e_C^2 + e_T'^2 - 2e_C e'_T \cos E_{C0}} \tag{11}$$

For use in the next section, it is helpful to note that the relative inclination γ can also be determined by parameters pertaining to the underlying inertial orbits of the target and chaser. It is shown in [17] that γ is the angle between the angular momentum vectors of the chaser's and target's orbits (see Fig. 2). In addition, the relative right ascension ϕ_0 can be shown to be

$$\phi_0 = E_{C0} + w \tag{12}$$

where w is the angle between the line of intersection between the chaser's and target's orbits and the eccentricity vector of the chaser's orbit, as shown in Fig. 2. Note that w is not defined when the two inertial orbits are coplanar. In that case, w can be defined with respect to the X_T axis.

Two-Burn Optimal Transfer

In the previous section, a parametrization of relative orbits was described, which allows for the expression of these orbits in terms of the offset y_0, minor axis δ, relative inclination γ, and relative right ascension ϕ_0. An optimal transfer problem between two such orbits can therefore be defined in terms of the initial and final values of δ, y_0, ϕ_0, and γ. In what follows, these values will be denoted by δ_1/δ_2, y_{01}/y_{02}, ϕ_{01}/ϕ_{02}, and γ_1/γ_2 on the initial/final orbits, respectively.

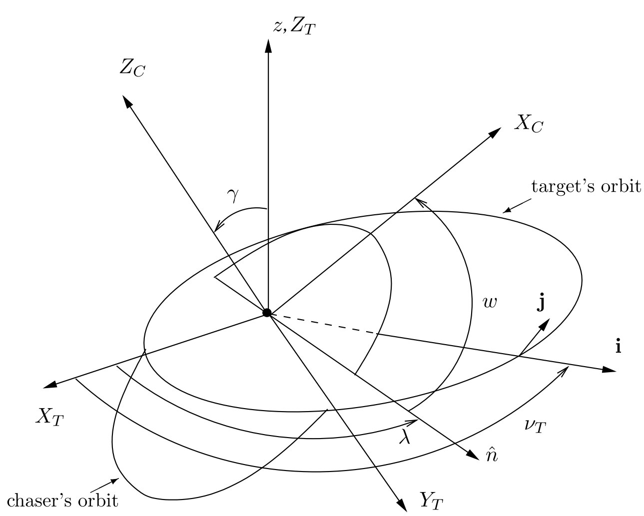

Fig. 2 Target's and chaser's orbits and associated coordinate systems.

Figure 3 shows a generic transfer between two relative orbits about a target in an (inertial) elliptic orbit, and is depicted in the target's LVLH coordinate system along with the required initial and final impulsive $\Delta\mathbf{v}$s. The main objective is therefore to minimize

$$|\Delta\mathbf{v}| = |(\Delta\mathbf{v})_1| + |(\Delta\mathbf{v})_2| \tag{13}$$

To obtain expressions for $(\Delta\mathbf{v})_1$ and $(\Delta\mathbf{v})_2$, we note that

$$(\Delta\mathbf{v})_1 = \mathbf{v}_{t0} - \mathbf{v}_{00} \tag{14}$$

$$(\Delta\mathbf{v})_2 = \mathbf{v}_{21} - \mathbf{v}_{t1} \tag{15}$$

where $\mathbf{v}_{t0}$ and $\mathbf{v}_{t1}$ are the relative velocities on the transfer trajectory at the beginning and the end of the transfer, respectively, $\mathbf{v}_{00}$ is the relative velocity on the initial orbit at the beginning of the transfer, and $\mathbf{v}_{21}$ is the relative velocity on the final orbit at the end of the transfer.

Frame Invariance of the Transfer

As has been stated earlier, the relative orbits under consideration are assumed to be about a target that, in its inertial motion, is in an elliptic orbit, of given eccentricity, about the center of force. However, in determining the required $|\Delta\mathbf{v}|$ and, in particular, the optimal $|\Delta\mathbf{v}|$ for a transfer between the relative orbits, it is possible to describe the transfer from the vantage point of a different reference frame. To that end, it is convenient in the current problem to consider the transfer as being carried out in the LVLH coordinate system of a target in *circular* orbit about the center of force, and with semimajor axis equal to that of the actual target. This new target will be referred to as the auxiliary target.

Using Eq. (11) with $e'_T = 0$ (for the circular auxiliary target orbit), note that the same inertial orbits that gave rise to the initial and final relative orbits in the reference frame of the actual target will give rise to relative orbits in the reference frame of the auxiliary target. These relative orbits will have semiminor axes

$$\delta'_1 = e_{C1} \tag{16}$$

and

$$\delta'_2 = e_{C2} \tag{17}$$

Clearly then, e_{C1} and e_{C2} must first be found. This can be done using Eqs. (2) and (7) for a given set of values of δ_1 and ϕ_{01}, on the initial orbit, and δ_2 and ϕ_{02} on the final orbit.

It is convenient to choose the auxiliary target's orbit in such a way that the values of y_0 on the initial and final orbits remain unchanged after the transformation. As y_0 is a parameter that is determined by the relative orientation of the chaser's and the target's inertial orbits (see [17]), its value remains unchanged in the auxiliary target's frame if the latter is chosen, so that its true anomaly $\nu_T = 0$ at the same time as for the actual target. Furthermore, if the auxiliary target's orbit is chosen so that it is coplanar with the orbit of the actual target, the relative inclinations γ_1 and γ_2 will remain unchanged after the transformation. This follows from the definition of γ in Fig. 2.

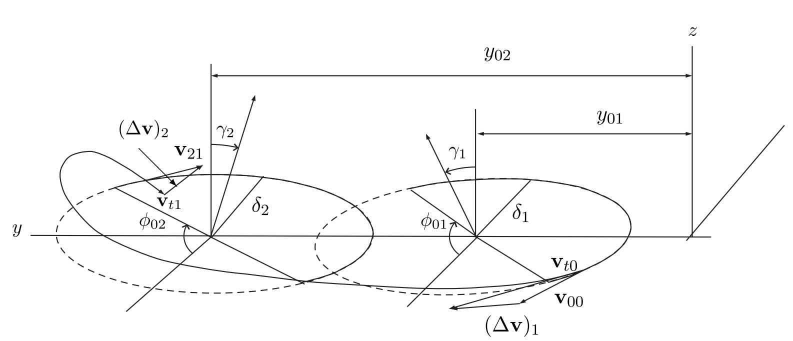

Fig. 3 Generic two-burn transfer between two relative orbits.

Next, the relative velocities depicted in Fig. 3 will correspond to new values in the reference frame of the auxiliary target. To see the relation between the relative velocities in the two reference frames (of the actual and auxiliary targets), one may start by noting that each relative velocity corresponds to a velocity of the target, and one of the chaser, in the inertial frame. For example, the inertial velocities for the chaser $\mathbf{V}_{00}$ and $\mathbf{V}_{t0}$ that correspond to $\mathbf{v}_{00}$ and $\mathbf{v}_{t0}$ in Fig. 3 are

$$\mathbf{V}_{00} = \mathbf{v}_T(\phi_1) + \mathbf{v}_{00} \tag{18}$$

and

$$\mathbf{V}_{t0} = \mathbf{v}_T(\phi_1) + \mathbf{v}_{t0} \tag{19}$$

where $\mathbf{v}_T(\phi_1)$ is the inertial velocity of the target (in its elliptic orbit about the center of force) at the instant of the initial burn, that is, $\phi = \phi_1$. It is therefore clear that another expression for the first impulsive velocity change $\Delta\mathbf{v}_1$ is

$$\Delta\mathbf{v}_1 = \mathbf{V}_{t0} - \mathbf{V}_{00} \tag{20}$$

On the other hand, the preceding inertial velocities can also be expressed in terms of the relative velocities with respect to the auxiliary target. Denoting these relative velocities as primed quantities

$$\mathbf{V}_{00} = \mathbf{v}_A(\phi_1) + \mathbf{v}'_{00} \tag{21}$$

and

$$\mathbf{V}_{t0} = \mathbf{v}_A(\phi_1) + \mathbf{v}'_{t0} \tag{22}$$

where $\mathbf{v}_A(\phi_1)$ is the velocity of the auxiliary target at the time of the first burn, that is, $\phi = \phi_1$. Defining

$$\Delta\mathbf{v}'_1 = \mathbf{v}'_{t0} - \mathbf{v}'_{00} \tag{23}$$

that is, the first velocity change in the reference frame of the auxiliary target, and using Eqs. (21) and (22), it is clear that

$$\Delta\mathbf{v}'_1 = \mathbf{V}_{t0} - \mathbf{V}_{00} \tag{24}$$

and, using Eq. (20),

$$\Delta\mathbf{v}_1 = \Delta\mathbf{v}'_1 \tag{25}$$

Thus, the first required impulsive $\Delta\mathbf{v}$ in the auxiliary target's reference frame is the same as what would be calculated in the actual target's reference frame.

A similar analysis for the relative velocities involved in the second burn leads to

$$\Delta\mathbf{v}_2 = \Delta\mathbf{v}'_2 \tag{26}$$

that is, also the second impulsive velocity change is independent of the reference frame used to describe the transfer.

In light of the preceding discussion, it can be concluded that the optimization of the transfer between the initial and final orbits in the given target's reference frame may equivalently be carried out in the auxiliary target's reference frame. This would of course prompt one to expect that the total $|\Delta\mathbf{v}|$ must not depend on the initial and final minor axes (i.e., δ_1 and δ_2) of the relative orbits, as these clearly change between reference frames. Indeed, it will be shown next that the total $|\Delta\mathbf{v}|$ only depends on the *difference* between these quantities and not on each value. The advantage of choosing a target in circular orbit for the new reference frame (the auxiliary target) is of course that analytical expressions for the $\Delta\mathbf{v}$s are readily available as the solution to the linearized Hill's equations.

Transfer in the Auxiliary Target's Reference Frame

It has been shown previously that an auxiliary target can be chosen so that, for the corresponding periodic relative orbits,

$$x = -\delta' \cos\phi \tag{27}$$

$$y = y_0 + 2\delta' \sin\phi \tag{28}$$

$$z = \gamma \sin(\phi - \phi_{z0}) \tag{29}$$

where $\delta' = e_C$ because the target is in circular orbit ($e'_T = 0$).

Therefore, the transfer is between a relative orbit of major axis $2\delta'_1$ to one of major axis $2\delta'_2$, and the goal is to determine the values of ϕ at the first and second $\Delta\mathbf{v}$s, ϕ_1 and ϕ_2 (see Fig. 3) that correspond to a minimized $|\Delta\mathbf{v}|$.

Denoting the positions of the first and second burns by $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, the initial velocity on the transfer orbit $\mathbf{v}_{t0}$ can be found from the requirement that, at the end of the transfer, at time t_f, the position should be

$$\mathbf{r}_2 = \Phi_{rr}(t_f)\mathbf{r}_1 + \Phi_{rv}(t_f)\mathbf{v}_{t0} \tag{30}$$

or

$$\mathbf{v}_{t0} = \Phi_{rv}^{-1}(\mathbf{r}_2 - \Phi_{rr}\mathbf{r}_1) \tag{31}$$

where Φ_{rr} and Φ_{rv} are the well-known Clohessy–Wiltshire matrices (see, e.g., [25,26]). Note that, because $n = 1$, the time of flight of transfer is

$$t_f = \phi_2 - \phi_1 - \phi_{02} - \phi_{01} \tag{32}$$

where it is assumed (without loss of generality) that the initial time, that is, at the first burn, is zero.

On the other hand, differentiating Eq. (30), the velocity on the initial orbit at the time of the first burn is known to be

$$\mathbf{v}_{00} = \delta'_1 \sin\phi_1 \mathbf{i} + 2\delta'_1 \cos\phi_1 \mathbf{j} + \gamma_1 \cos(\phi_1 - \phi_{z01})\mathbf{k} \tag{33}$$

and the first impulsive velocity change may be written

$$(\Delta\mathbf{v})_1 = \mathbf{v}_{t0} - \mathbf{v}_{00} \tag{34}$$

In the same way, the second impulsive velocity change will be

$$(\Delta\mathbf{v})_2 = \mathbf{v}_{21} - \mathbf{v}_{t1} \tag{35}$$

where

$$\mathbf{v}_{t1} = \Phi_{vr}(t_f)\mathbf{r}_1 + \Phi_{vv}(t_f)\mathbf{v}_{t0} \tag{36}$$

is the velocity on the transfer orbit at the time of the second burn t_f, and where $\Phi_{vr}(t_f)$ and $\Phi_{vv}(t_f)$ are the Clohessy–Wiltshire matrices for the relative velocity. Also,

$$\mathbf{v}_{21} = \delta'_2 \sin\phi_2 \mathbf{i} + 2\delta'_2 n \cos\phi_2 \mathbf{j} + \gamma_2 \cos(\phi_2 - \phi_{z02})\mathbf{k} \tag{37}$$

is the velocity on the final orbit at the time of the second burn.

Role of the Time of Transfer

Considering Eqs. (34) and (35), it is clear that the total $|\Delta\mathbf{v}|$ will depend on the positions and corresponding velocities on the initial and final orbits at the times of the first and second burns, respectively. In addition, the total $\Delta\mathbf{v}$ will depend on the time between the first and the second burns. It is shown in the Appendix that, in the case where the initial and final relative orbits are in parallel planes, that is, $\gamma_1 = \gamma_2$ and $\phi_{z01} = \phi_{z02}$, the total $|\Delta\mathbf{v}|$ has a minimum when the time of flight is chosen to be equal to $\phi_2 - \phi_1$, that is, when the initial and final orbits have the same phase values or

$$\phi_{01} = \phi_{02} \tag{38}$$

It is of course true that the transfer that is thus obtained corresponds to a local minimum of $|\Delta\mathbf{v}|$ and not necessarily a global one. Nevertheless, transfers with values of $|\Delta\mathbf{v}|$ smaller than the one obtained will require larger transfer times.

Expression for the Total $|\Delta \mathbf{v}|$

Based on the results of the previous section, it will be assumed in what follows that the time of transfer is

$$t_1 - t_1 = \phi_2 - \phi_1 \tag{39}$$

and that

$$\gamma_1 = \gamma_2 \tag{40}$$

Then, using Eq. (34) and making straightforward but significant simplifications shows that

$$(\Delta \mathbf{v})_1 = \begin{bmatrix} b_{11} & -b_{11} & c_{11} & -c_{11} \\ b_{21} & -b_{21} & c_{21} & -c_{21} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \delta_1' \\ \delta_2' \\ y_{01} \\ y_{02} \end{pmatrix} \tag{41}$$

where

$$b_{11} = \frac{3(\phi_1 - \phi_2)\cos\phi_2 - 4\sin\phi_1 + 4\sin\phi_2}{-8 + 8\cos(\phi_1 - \phi_2) + 3(\phi_1 - \phi_2)\sin(\phi_1 - \phi_2)} \tag{42}$$

$$b_{21} = \frac{-2\cos\phi_1 + 2\cos\phi_2}{-8 + 3(\phi_1 - \phi_2)\cos\phi_2\sin\phi_1 + 8\sin\phi_1\sin\phi_2 + \cos\phi_1[8\cos\phi_2 + 3(-\phi_1 + \phi_2)\sin\phi_2]} \tag{43}$$

$$c_{11} = \frac{-2 + 2\cos(\phi_1 - \phi_2)}{-8 + 8\cos(\phi_1 - \phi_2) + 3(\phi_1 - \phi_2)\sin(\phi_1 - \phi_2)} \tag{44}$$

$$c_{21} = \frac{\sin(\phi_1 - \phi_2)}{-8 + 3(\phi_1 - \phi_2)\cos\phi_2\sin\phi_1 + 8\sin\phi_1\sin\phi_2 + \cos\phi_1[8\cos\phi_2 + 3(-\phi_1 + \phi_2)\sin\phi_2]} \tag{45}$$

and using Eq. (35) now gives (after simplification)

$$(\Delta \mathbf{v})_2 = \begin{bmatrix} b_{12} & -b_{12} & c_{12} & -c_{12} \\ b_{22} & -b_{22} & c_{22} & -c_{22} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \delta_1' \\ \delta_2' \\ y_{01} \\ y_{02} \end{pmatrix} \tag{46}$$

where

$$b_{12} = \frac{-3(\phi_1 - \phi_2)\cos\phi_1 + 4\sin\phi_1 - 4\sin\phi_2}{-8 + 8\cos(\phi_1 - \phi_2) + 3(\phi_1 - \phi_2)\sin(\phi_1 - \phi_2)} \tag{47}$$

$$b_{22} = \frac{2\cos\phi_1 - 2\cos\phi_2}{-8 + 3(\phi_1 - \phi_2)\cos\phi_2\sin\phi_1 + 8\sin\phi_1\sin\phi_2 + \cos\phi_1[8\cos\phi_2 + 3(-\phi_1 + \phi_2)\sin\phi_2]} \tag{48}$$

$$c_{12} = \frac{-2 + 2\cos(\phi_1 - \phi_2)}{-8 + 8\cos(\phi_1 - \phi_2) + 3(\phi_1 - \phi_2)\sin(\phi_1 - \phi_2)} \tag{49}$$

$$c_{22} = \frac{\cos\phi_2\sin\phi_1 - \cos\phi_1\sin\phi_2}{-8 + 3(\phi_1 - \phi_2)\cos\phi_2\sin\phi_1 + 8\sin\phi_1\sin\phi_2 + \cos\phi_1[8\cos\phi_2 + 3(-\phi_1 + \phi_2)\sin\phi_2]} \tag{50}$$

Using Eqs. (41) and (46), it is then easy to verify that the magnitudes of the impulsive velocity changes are

$$|(\Delta \mathbf{v})_1| = |\Delta\delta|\sqrt{\mathbf{b}_1^2 + 2\mathbf{b}_1 \cdot \mathbf{c}_1\rho + \mathbf{c}_1^2\rho^2} \tag{51}$$

where

$$\Delta\delta = \delta_2' - \delta_1' \tag{52}$$

$$\rho = \frac{\Delta y_0}{\Delta\delta} \qquad (\Delta\delta \neq 0) \tag{53}$$

with

$$\Delta y_0 = y_{02} - y_{01} \tag{54}$$

and where, for the simplicity of the expression, the following vectors have been defined:

$$\mathbf{b}_1 = (b_{11}, b_{21})^T \tag{55}$$

$$\mathbf{c}_1 = (c_{11}, c_{21})^T \tag{56}$$

so that

$$\mathbf{b}_1^2 = b_{11}^2 + b_{21}^2 \tag{57}$$

$$\mathbf{c}_1^2 = c_{11}^2 + c_{21}^2 \tag{58}$$

$$\mathbf{b}_1 \cdot \mathbf{c}_1 = b_{11}c_{11} + b_{21}c_{21} \tag{59}$$

Clearly, $\Delta\delta \neq 0$ must be assumed in Eq. (51); the case $\Delta\delta = 0$ will be considered in a later section. In the same way,

$$|(\Delta\mathbf{v})_2| = |\Delta\delta|\sqrt{\mathbf{b}_2^2 + 2\mathbf{b}_2 \cdot \mathbf{c}_2\rho + \mathbf{c}_2^2\rho^2} \tag{60}$$

where

$$\mathbf{b}_2 = (b_{12}, b_{22})^T \tag{61}$$

$$\mathbf{c}_2 = (c_{12}, c_{22})^T \tag{62}$$

$$\mathbf{b}_2^2 = b_{12}^2 + b_{22}^2 \tag{63}$$

$$\mathbf{c}_2^2 = c_{12}^2 + c_{22}^2 \tag{64}$$

$$\mathbf{b}_2 \cdot \mathbf{c}_2 = b_{12}c_{12} + b_{22}c_{22} \tag{65}$$

Now, considering Eqs. (51), (60), and (13), it follows that

$$|\Delta\mathbf{v}| = |\Delta\delta|\eta(\phi_1, \phi_2, \rho) \tag{66}$$

where

$$\eta(\phi_1, \phi_2, \rho) = \sqrt{\mathbf{b}_1^2 + 2\mathbf{b}_1 \cdot \mathbf{c}_1\rho + \mathbf{c}_1^2\rho^2} + \sqrt{\mathbf{b}_2^2 + 2\mathbf{b}_2 \cdot \mathbf{c}_2\rho + \mathbf{c}_2^2\rho^2} \tag{67}$$

Thus, it is clear that the total $|\Delta\mathbf{v}|$ is directly proportional to the difference in the x amplitude δ between initial and final orbits, and also depends on the ratio of Δy to $\Delta\delta$.

Numerical Analysis

It would now be possible to differentiate the expression for the total $|\Delta\mathbf{v}|$ to solve for the values of ϕ_1 and ϕ_2 that render the derivative zero. These values would correspond to a minimum value for $|\Delta\mathbf{v}|$. However, the resulting expressions would be of a transcendental nature in ϕ_1 and ϕ_2, and therefore an analytical solution would not be available; any solution would have to be obtained via numerical methods.

On the other hand, if one is to use numerical methods to obtain the solution, it is just as convenient to numerically find the minima of $\eta(\phi_1, \phi_2, \rho)$ directly and as functions of the parameter ρ. In what follows, these numerical calculations are performed through the use of functions available in Mathematica [27].

Case of Coplanar Orbits

As a starting point for the numerical investigations, the case where the initial and final orbits are concentric and coplanar is studied in this section. Note then that $\rho = 0$. Therefore, for the sake of brevity, one can define a new function through

$$\eta_0(\phi_1, \phi_2) = \eta(\phi_1, \phi_2, 0) \tag{68}$$

Figure 4 shows a contour plot of the function $\eta_0(\phi_1, \phi_2)$ for the range of values of $\phi_1 \in (-\pi, \pi)$ and $\phi_2 \in (0, 2\pi)$. There are in this figure four local minima, labeled A_0, B_0, C_0, and D_0. The value of $\eta_0(\phi_1, \phi_2)$ at these local minima has been determined numerically to be 0.860032. The minima are at A_0: $(\phi_1, \phi_2) = (-3.02248, 1.21415)$, B_0: $(\phi_1, \phi_2) = (-1.21415, 3.02248)$, C_0: $(\phi_1, \phi_2) = (0.119111, 4.35574)$, and D_0: $(\phi_1, \phi_2) = (1.92744, 6.16407)$.

It may be noted that the minima lie on lines $\phi_1 + \phi_2 =$ constant and at values of $\Delta\phi = \phi_2 - \phi_1$ symmetrically located about $\Delta\phi = 0$.

The two minimum-$|\Delta\mathbf{v}|$ transfers in the LVLH frame that correspond to the minima A_0 and B_0 in Fig. 4 are exemplified in Fig. 5, where the initial and final orbits are taken to be such that $\delta_1 = 0.001$ and $\delta_2 = 0.002$. In the same way, Fig. 6 shows examples of transfers corresponding to points C_0 and D_0 in Fig. 4, where again $\delta_1 = 0.001$ and $\delta_2 = 0.002$.

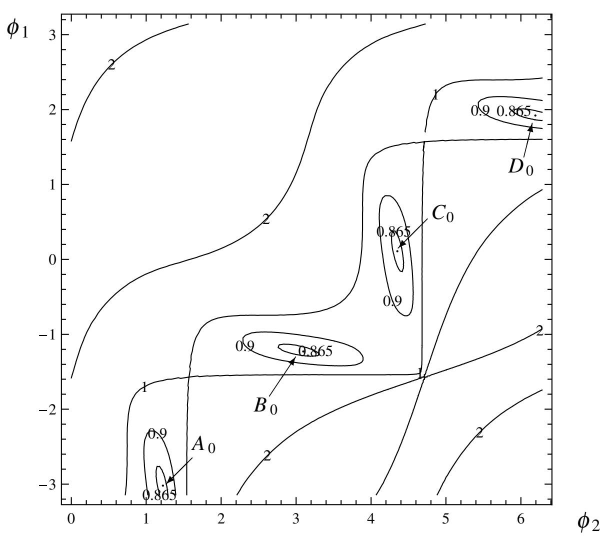

Fig. 4 Contour plot of $\eta_0(\phi_1, \phi_2)$.

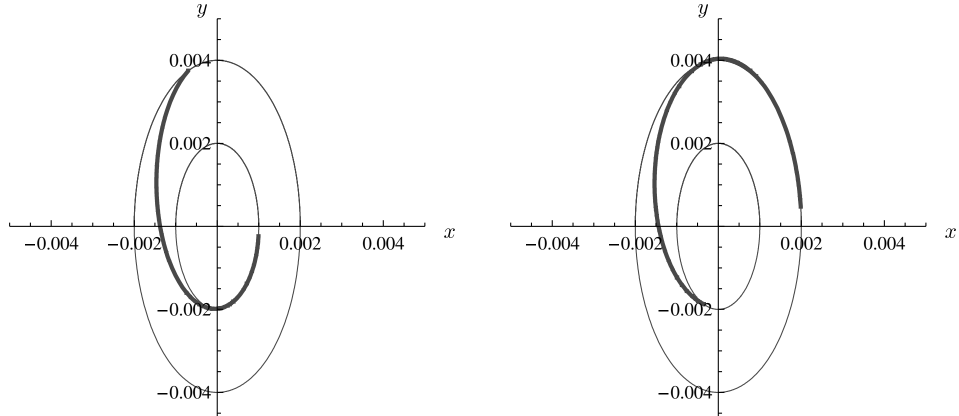

Fig. 5 Optimal transfer orbits (thick lines) corresponding to points A_0 and B_0 in Fig. 4. Initial orbit has $\delta_1 = 0.001$ and final orbit has $\delta_2 = 0.002$.

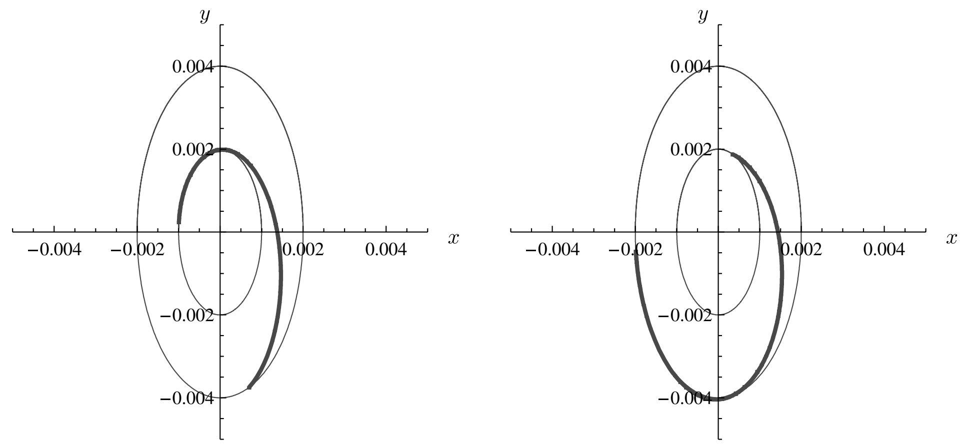

Fig. 6 Optimal transfer orbits (thick lines) corresponding to points C_0 and D_0 in Fig. 4. Initial orbit has $\delta_1 = 0.001$ and final orbit has $\delta_2 = 0.002$.

Relative Orbits in Parallel Planes: $\rho \neq 0$

In the case where ρ is not zero, that is, the initial and final relative orbits are not concentric, but still have the same inclination and right ascension, the optimal transfer between relative orbits can again be obtained numerically for varying values of ρ. As ρ varies, the minima of $\eta(\phi_1, \phi_2, \rho)$ trace out paths in the ϕ_1–ϕ_2 plane, which are shown in Fig. 7 for ρ in the range $(-5, 5)$. In further explaining the figure, it is convenient to refer to the curves that represent the evolution of the local minima for varying values of ρ by $A(\rho)$, $B(\rho)$, $C(\rho)$, and $D(\rho)$, such that $A(\rho)$ is the path that goes through A_0 at $\rho = 0$, $B(\rho)$ is the path that goes through B_0 at $\rho = 0$, etc. Then, one can first verify that the points A_0, B_0, C_0, and D_0 are the same points as the minima in Fig. 4.

Case $\rho > 0$

When the value of ρ is increased from zero, points $B(\rho)$ and $C(\rho)$ will travel toward the points labeled $B_{2^-} = B(2^-)$ and $C_{2^-} = C(2^-)$ (i.e., these points correspond to ρ approaching a value of 2 from below). It is worth noting that $\rho = 2$ means that initial and final orbits intersect at $\phi = \pi$ rad. The local minima $B(\rho)$ and $C(\rho)$ cease to exist for values of $\rho \geqq 2$. This is illustrated in Fig. 8, which shows a contour plot of $\eta(\phi_1, \phi_2, \rho)$ at $\rho = 2$. The location of B_{2^-} and C_{2^-} are approximated by B_{2^-}: $(\phi_1, \phi_2) = (-1.571, 3.00)$ and C_{2^-}: $(\phi_1, \phi_2) = (0.0, 4.71)$, respectively. These values are calculated for $\rho = 1.999$.

As would be expected, the value of $\eta(\phi_1, \phi_2, \rho)$ at the local minima $B(\rho)$ and $C(\rho)$ is itself a function of ρ. Numerical calculations show that, for a given ρ, this value is the same on $B(\rho)$ and $C(\rho)$, and may therefore be denoted $(\eta_{\min})_{BC}(\rho)$. Figure 9 shows the dependence of $(\eta_{\min})_{BC}(\rho)$ on ρ. Note that $(\eta_{\min})_{BC}(2^-) \approx 1$.

On the other hand, points $A(\rho)$ and $D(\rho)$ trace out their own curves in the ϕ_1, ϕ_2 plane as ρ increases from zero. Thus, the two points start at A_0 and D_0 and travel along their respective paths until they reach $A_m = A(1.63)$ and $D_m = D(1.63)$. A further increase of ρ causes the local minima to travel toward the points labeled $A_r = A(1.85)$ and $D_r = D(1.85)$, respectively. Increasing ρ past the value of 1.85 causes the local minima to now reverse direction and move back toward and past the points $A_m = A(2.13)$ and $D_m = D(2.13)$, respectively.

As in the case of the local minima $B(\rho)$ and $C(\rho)$, the value of $\eta(\phi_1, \phi_2, \rho)$ at the local minima $A(\rho)$ and $D(\rho)$ is itself dependent on

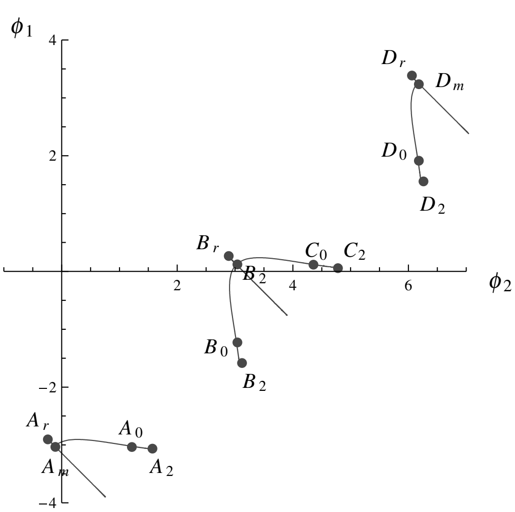

Fig. 7 Dependence of local minima of $\eta(\phi_1, \phi_2, \rho)$ on ρ.

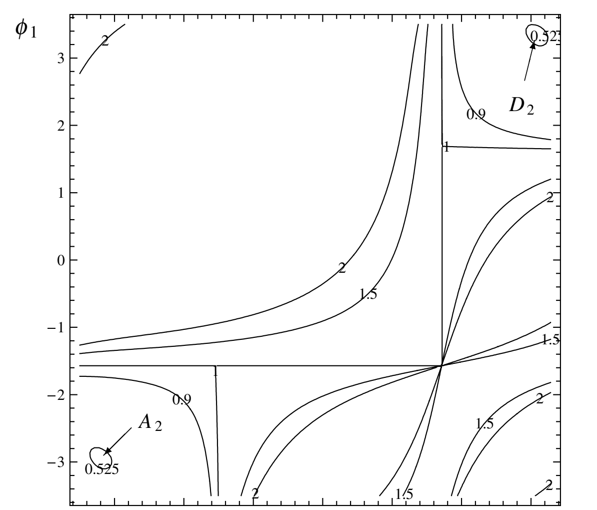

Fig. 8 Contour plot of $\eta(\phi_1, \phi_2, 2)$.

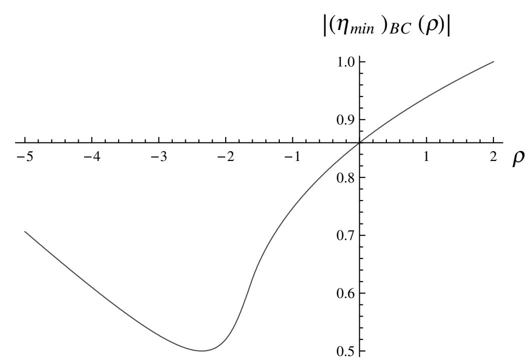

Fig. 9 Dependence of minimum $|\Delta v|$ on ρ for points B and C.

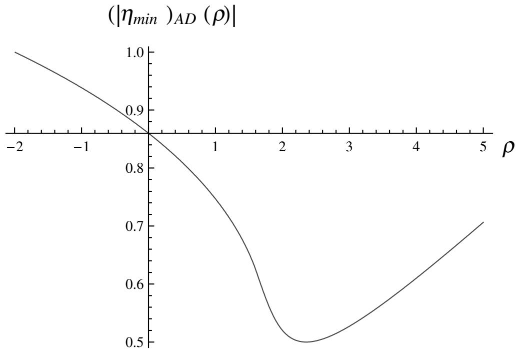

Fig. 10 Dependence of minimum $|\Delta v|$ on ρ for points A and D.

ρ, and has the same value at the two points $A(\rho)$ and $D(\rho)$ for a given ρ. Denoting this minimum value by $(\eta_{\min})_{AD}(\rho)$, Fig. 10 shows its dependence on ρ in the interval $\rho \in (-2, 5)$. Note that $|(\eta_{\min})_{AD}(\rho)|$ itself has a minimum at $\rho \approx 2.2$ with a value of $\eta_{\min} = 0.5$.

Case $\rho < 0$

A decrease in ρ from $\rho = 0$ causes the local minima to trace their respective curves in Fig. 7 in the opposite direction to that described for the case of $\rho > 0$. First, considering points $A(\rho)$ and $D(\rho)$, these points will travel from A_0 and B_0 at $\rho = 0$ toward the points $A_{-2^+} = A(-2^+)$ and $D_{-2^+} = D(-2^+)$, respectively. The local minima $A(\rho)$ and $D(\rho)$ cease to exist for $\rho \leqq -2$. Note that $\rho = -2$ corresponds to the case where the initial and final orbits intersect at $\phi = 0$. The location of the critical points are approximated by A_{-2^+}: $(\phi_1, \phi_2) = (-3.00, 1.571)$ and $D_{-2^+} = (\phi_1, \phi_2) = (1.571, 5.99)$, respectively,

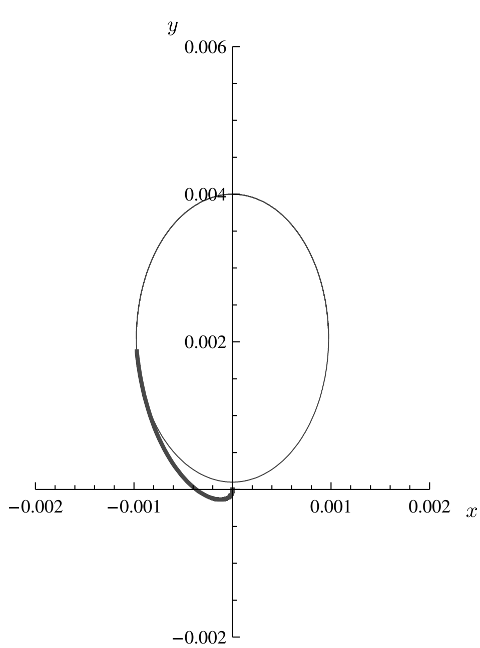

Fig. 11 Optimal deployment of relative satellite in the target's reference frame.

which are calculated at $\rho = -1.999$. At both these points, the corresponding value of the function $\eta(\phi_1, \phi_2, 1.999) \approx 1$.

For a given ρ, $\eta(\phi_1, \phi_2, \rho)$ has the same value at the local minima $A(\rho)$ and $D(\rho)$, which may therefore be denoted $(\eta_{\min})_{AD}(\rho)$. Figure 10 shows the dependence of $|(\eta_{\min})_{AD}(\rho)|$ on ρ in the interval $\rho \in (-2, 5)$, and, in particular, how a decrease in ρ increases $|(\eta_{\min})_{AD}(\rho)|$.

On the other hand, points $B(\rho)$ and $C(\rho)$ trace out their own curves in the ϕ_1–ϕ_2 plane for ρ decreasing from a value of zero. Thus, the two points start at B_0 and C_0 and travel along their respective paths until they reach $B_m = B(-1.63)$ and $C_m = C(-1.63)$, where they merge. A further decrease of ρ causes the now single local minimum to travel toward the point labeled $B_r = B(-1.85)$. Decreasing ρ further causes the local minimum to now reverse direction and move back toward and past the point $B_m = B(-2.13)$.

For a given ρ, $\eta(\phi_1, \phi_2, \rho)$ has the same value at the local minima $B(\rho)$ and $C(\rho)$, which may therefore be denoted $(\eta_{\min})_{BC}(\rho)$. Figure 9 shows the dependence of $|(\eta_{\min})_{BC}(\rho)|$ on ρ in the interval $\rho \in (-5, 2)$. Note that $|(\eta_{\min})_{BC}(\rho)|$ itself has a minimum value of 0.5 at $\rho \approx -2.2$.

Example

It was noted in the previous sections that $|(\eta_{\min})_{AD}(\rho)|$ and $|(\eta_{\min})_{BC}(\rho)|$ (and therefore the corresponding $|\Delta \mathbf{v}|$s) are smallest at $\rho = 2.2$ and $\rho = -2.2$, respectively. This result can be directly used to advantage in problems where ρ is a parameter that can be varied.

As an example, consider a situation where a relative satellite is to be deployed from a spacecraft in an elliptic orbit with a requirement on the relative orbit that the largest distance from the launching spacecraft (the target) to the relative satellite (the chaser) is a given value ad, where a is the semimajor axis of the target's orbit, and that

$$\phi_0 = 0 \tag{69}$$

Then, Fig. 9 suggests that there is a value of δ (i.e., amplitude of the relative orbit) that would correspond to $\rho = 2.2$ or $\rho = -2.2$, and that would therefore be most economical in terms of the required $|\Delta \mathbf{v}|$. To determine this value, first note that, because the satellite is being deployed from the target, $\delta_1 = y_{01} = 0$, that is, in the reference frame of the target, the initial semiminor axis and center offset are zero. The requirement on the relative orbit is then (in terms of variables defined in the target's frame)

$$d = y_{02} + 2\delta_2 \tag{70}$$

or, using Eq. (2),

$$d = y_{02} + 2\sqrt{e_C^2 + e_T^2 - 2e_C e_T \cos E_{C0}} \tag{71}$$

The change in the position of the center of the orbit is therefore

$$\Delta y = y_{02} - y_{01} = y_{02} \tag{72}$$

Therefore, using Eq. (53),

$$\Delta y = y_{02} = \rho \Delta \delta \tag{73}$$

Here, $\Delta\delta$, defined in Eq. (52), is the difference in amplitude between the initial and final relative orbits in the auxiliary target's frame. Using Eqs. (7) and (69), it follows that $E_{C0} = 0$ and Eq. (2) gives

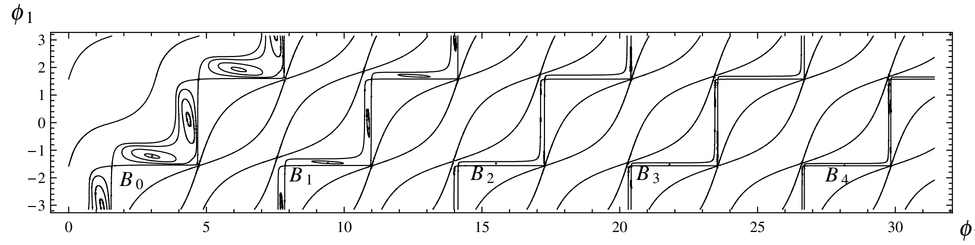

Fig. 12 Contour plot of $\eta(\phi_1, \phi_2, 0)$.

Table 1 Set of local minima of $\eta(\phi_1, \phi_2, 0)$ depicted in Fig. 12

	Local minima of $\eta(\phi_1, \phi_2, 0)$		
	ϕ_1	ϕ_2	$\eta(\phi_1, \phi_2, 0)$
B_0	−1.214	3.022	0.8600
B_1	−1.442	9.388	0.9406
B_2	−1.491	15.686	0.9620
B_3	−1.512	21.98	0.9720
B_4	−1.525	28.26	0.9779

$$\delta_1' = e_T \tag{74}$$

$$\delta_2' = e_C \tag{75}$$

and it follows that

$$\Delta\delta = \delta_2' - \delta_1' = e_C - e_T \tag{76}$$

This, together with Eq. (73), now gives

$$y_{02} = \rho(e_C - e_T) \tag{77}$$

and, using Eq. (71) with Eq. (69),

$$d = \rho(e_C - e_t) + 2\sqrt{(e_C - e_T)^2} \tag{78}$$

or, defining $\delta = e_C - e_T$,

$$d = \rho\delta + 2|\delta| \tag{79}$$

Solving for δ with $\rho = 2.2$ now gives

$$\delta = d/4.2 \tag{80}$$

An implementation of the corresponding transfer for the local minimum at $D(2.2)$ gives $\phi_1 = 3.233$ and $\phi_2 = 6.19$ and is shown in Fig. 11 where $d = 0.001$.

Large Transfer Angles

In studying the local minima of $|\eta(\phi_1, \phi_2, \rho)|$ numerically in the previous sections, it was necessary to limit the range of values considered for ϕ_1, ϕ_2, and ρ. The intervals $\phi_1 \in (-\pi, \pi)$ and $\phi_2 \in (0, 2\pi)$ were chosen with the aim of considering transfers of $\Delta\phi = \phi_2 - \phi_1 < 2\pi$, as a practical first step.

It is, however, of interest to consider larger values of $\Delta\phi$ to discern the effect of larger transfer angles on the required $|\Delta\mathbf{v}|$. In doing so, it is convenient to hold ϕ_1 within the range $(-\pi, \pi)$ while letting ϕ_2 increase to larger values, without loss of generality. Thus, as ϕ_2 increases, the overall structure of the contours of the function $\eta(\phi_1, \phi_2, \rho)$ in ϕ_1–ϕ_2, and in particular the "staircase" structure enclosing the local minima, is repeated as shown in Fig. 12. However, it is clear that the widths of consecutive staircases enclosing the local minima decrease for increasing values of ϕ_2. This suggests that the values of Δv at the enclosed local minima should approach the value of 1 (the value on the staircase), barring a discontinuity at that point. Indeed, this is easily verified for the local minima. As an example, starting with the local minimum B_0, consider the horizontal row of local minima B_1–B_4 in Fig. 12. The exact locations of these minima and the corresponding value of $\eta_{\min}$ are given in Table 1.

Thus, while the value of ϕ_2 increases, the value of $|\eta_{\min}|$ is increasing and approaching unity. A similar result is obtained for the other local minima A, C, and D.

Dependence of the Local Minima on ρ

As in the case of smaller transfer angles, $\Delta\phi < 2\pi$, the locations of the local minima and corresponding value of $\Delta\mathbf{v}$ for larger values of $\Delta\phi$ will generally depend on the value of ρ.

This dependence can be studied numerically in just the same way as was done for the case of $\Delta\phi < 2\pi$ and results in diagrams that trace out the positions of the local minima in the ϕ_1–ϕ_2 plane, similar to those in Fig. 7. However, the value of ρ for which the various critical points on the diagrams are reached are generally different from those in the case of the minima $A(\rho)$, $B(\rho)$, $C(\rho)$, and $D(\rho)$, which were considered earlier. Also, it is important to note that the value of ρ, where $\eta(\phi_1, \phi_2, \rho)$ is minimum along a certain path, depends on the specific path. For example, if one considers the local minimum $B_1(\rho)$ and the corresponding minimum value of $|\eta(\phi_1, \phi_2, \rho)|$, one obtains Fig. 13 (solid curve) where the corresponding diagram for local minimum $B_0(\rho)$ is shown in the same diagram (dashed curve). The conclusion from this diagram is that if $\rho < 5$ (approximately), that is, if the distance between the centers of the initial and final orbits is not larger than roughly 5 times the difference in their semimajor axes, then the smaller $\Delta\phi$ corresponding to $B_0(\rho)$ is more efficient than the longer one corresponding to $B_1(\phi)$.

Pure Shift of Orbits

One possible type of transfer of relative orbits is where the size and orientation of the orbits remains unchanged while the center is shifted along the y axis, that is, the case where $\Delta\delta = 0$ and $\Delta y \neq 0$. The requirement $\Delta\delta = 0$ makes it necessary to redefine $\eta(\phi_1, \phi_2, \rho)$, as the variable ρ is no longer defined. To this end, using Eqs. (41) and (46) with Eq. (13), and keeping in mind that $\delta_1' = \delta_2'$, gives

$$|\Delta\mathbf{v}| = |\Delta y|\left(\sqrt{c_{11}^2 + c_{21}^2}\right) + \sqrt{c_{12}^2 + c_{22}^2} \tag{81}$$

Substituting the expressions for c_{11}, c_{21}, c_{12}, and c_{22} and simplifying gives

$$|\Delta\mathbf{v}| = |\Delta y|\eta_s(\Delta\phi) \tag{82}$$

where

$$\Delta\phi = \phi_2 - \phi_1 \tag{83}$$

and where

$$\eta_s(\Delta\phi) = \sqrt{\frac{10 - 6\cos\Delta\phi}{(3\Delta\phi\cos\Delta\phi - 8\sin\Delta\phi)^2}} \tag{84}$$

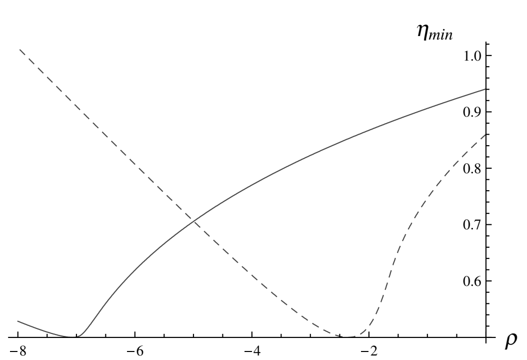

Fig. 13 Comparison of minimum $\eta(\phi_1, \phi_2, \rho)$ on local minima $B_0(\rho)$ (dashed) and $B_1(\rho)$ (solid).

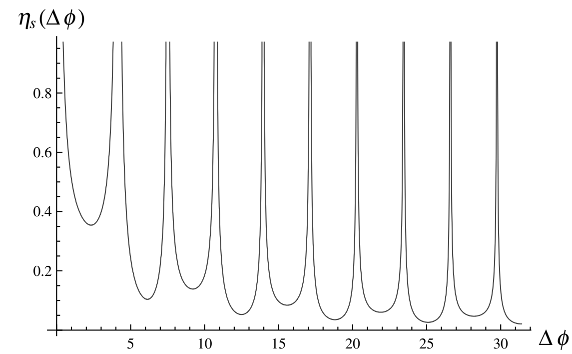

Fig. 14 Plot of $\eta_s(\Delta\phi)$ in the case where the change in δ between the initial and final orbits is zero.

$$
\begin{aligned}
|\Delta\mathbf{v}| = \Bigg\{ & \frac{((y_{01}-y_{02})\cos(\frac{t}{2}) + 2\{e_2\sin[\frac{1}{2}(t-2u-v)] + e_1\sin[\frac{1}{2}(t+2u-v)]\})^2}{[3t\cos(\frac{t}{2}) - 8\sin(\frac{t}{2})]^2} \\
& + \frac{1}{[8\cos(t)+3t\sin(t)-8]^2}\left[2y_{01} - 2y_{02} - 2(y_{01}-y_{02})\cos(t) - 3e_2 t\cos\left(t-u-\frac{v}{2}\right) + 3e_1 t\cos\left(u-\frac{v}{2}\right)\right. \\
& \left. + 4e_2\sin\left(t-u-\frac{v}{2}\right) + 4e_1\sin\left(u-\frac{v}{2}\right) - 4e_1\sin\left(t+u-\frac{v}{2}\right) + 4e_2\sin\left(u+\frac{v}{2}\right)\right]^2 \\
& + \csc^2(t)\left[\gamma_1\sin\left(u-\frac{v}{2}+w_{01}\right) + \gamma_2\sin\left(t-u-\frac{v}{2}-w_{02}\right)\right]^2\Bigg\}^{\frac{1}{2}} \\
& + \Bigg\{\frac{4\sin^2(\frac{t}{2})((y_{01}-y_{02})\cos(\frac{t}{2}) + 2\{e_2\sin[\frac{1}{2}(t-2u-v)] + e_1\sin[\frac{1}{2}(t+2u-v)]\})^2}{[8\cos(t)+3t\sin(t)-8]^2} \\
& + \frac{1}{[8\cos(t)+3t\sin(t)-8]^2}\left[-2y_{01} + 2y_{02} + 2(y_{01}-y_{02})\cos(t) + 3e_1 t\cos\left(t+u-\frac{v}{2}\right)\right. \\
& - 3e_2 t\cos\left(u+\frac{v}{2}\right) + 4e_2\sin\left(t-u-\frac{v}{2}\right) + 4e_1\sin\left(u-\frac{v}{2}\right) - 4e_1\sin\left(t+u-\frac{v}{2}\right) \\
& \left. + 4e_2\sin\left(u+\frac{v}{2}\right)\right]^2 + \csc^2(t)\left[\gamma_1\sin\left(t+u-\frac{v}{2}+w_{01}\right) - \gamma_2\sin\left(u+\frac{v}{2}+w_{02}\right)\right]^2\Bigg\}^{\frac{1}{2}}
\end{aligned} \tag{A3}
$$

has been defined. Thus $|\Delta\mathbf{v}|$, in the case where there is no change in amplitude of the relative orbit, only depends on the angular displacement.

Figure 14 is a plot of $\eta_s(\Delta\phi)$ as a function of $\Delta\phi$. It is clear from the figure that the value of $\eta_s(\Delta\phi)$ at consecutive local minima will alternately increase and decrease with increasing $\Delta\phi$. However, the overall tendency (for large increases in $\Delta\phi$) is for $\eta_s(\Delta\phi)$ to decrease.

Conclusions

In this paper, it has been shown that transfers between relative orbits with respect to a target in elliptic orbit can be studied through a convenient transformation of the problem to the reference frame of an auxiliary target in circular orbit. The method simplifies the problem significantly by allowing the use of the well known Clohessy–Wiltshire solutions to the problem of linear relative motion. The transformation itself is based on the expression of the relative motion between satellites as the difference between their respective motion about the center of force, in the inertial frame. This is in contrast to studying the relative motion as a solution to equations of motion obtained in the reference frame of the target.

Optimal transfers between the relative orbits have been characterized in this paper by the values of the amplitude (size), center location (offset), relative right ascension, and relative inclination, of the initial and final relative orbits. In the case of concentric relative orbits, four optimal transfers, all corresponding to the same value of the total $|\Delta\mathbf{v}|$, for a given difference in minor axes, have been identified. It is further shown that this value of $|\Delta\mathbf{v}|$ changes continuously with the distance between the centers of the initial and final orbits, and that it has a minimum when the distance between the centers is 2.2 times the difference in the minor axes.

Appendix: Optimal Time of Transfer

The total $|\Delta\mathbf{v}|$ is the sum of the magnitudes of the two impulsive velocity changes given in Eqs. (34) and (35). These expressions are in terms of the value of the angle ϕ at the times of the initial and final burns ϕ_1 and ϕ_2 and the time of flight. It is convenient to define two new variables through

$$u = \tfrac{1}{2}(\phi 1 + \phi 2) \tag{A1}$$

$$v = \phi 2 - \phi 1 \tag{A2}$$

In terms of u and v, the expression for $|\Delta v|$ becomes

Now, for a minimum of $|\Delta\mathbf{v}|$, a necessary condition is that the derivatives of $|\Delta\mathbf{v}|$ with respect to the variables u, v, and t vanish. Obtaining these derivatives is straightforward, however, tedious. It can be shown that, under the assumption $\gamma_1 = \gamma_2$ and $w_{01} = w_{02} = 0$,

$$\frac{\partial|\Delta\mathbf{v}|}{\partial u} = (e_1 - e_2)g(u,t) \tag{A4}$$

$$\frac{\partial|\Delta\mathbf{v}|}{\partial v} = (e_1 + e_2)g(u,t) \tag{A5}$$

for $t = v$ and where

$$
\begin{aligned}
g(u,t) = & \Bigg\{8\cos(u)\sin\left(\frac{t}{2}\right)^2\left[(y_{01}-y_{02})\cos\left(\frac{t}{2}\right)\right. \\
& \left. + 2(e_1-e_2)\sin(u)\right] + \left[-2y_{01} + 2y_{02}\right. \\
& + 2(y_{01}-y_{02})\cos(t) + 3(e_1-e_2)t\cos\left(\frac{t}{2}+u\right) \\
& - 4e_1\sin\left(\frac{t}{2}-u\right) + 4e_2\sin\left(\frac{t}{2}-u\right) - 4e_1\sin\left(\frac{t}{2}+u\right) \\
& \left. + 4e_2\sin\left(\frac{t}{2}+u\right)\right]\left[4\cos\left(\frac{t}{2}-u\right) - 4\cos\left(\frac{t}{2}+u\right)\right. \\
& \left. - 3t\sin\left(\frac{t}{2}+u\right)\right]\Bigg\} \Big/ (-8 + 8\cos(t) \\
& + 3t\sin(t))^2 \Big/ \left(\Bigg\{4\sin\left(\frac{t}{2}\right)^2\left[(y_{01}-y_{02})\cos\left(\frac{t}{2}\right)\right.\right. \\
& \left. + 2(e_1-e_2)\sin(u)\right]^2 + \left[-2y_{01} + 2y_{02} + 2(y_{01}-y_{02})\cos(t)\right. \\
& + 3(e_1-e_2)t\cos\left(\frac{t}{2}+u\right) - 4e_1\sin\left(\frac{t}{2}-u\right) \\
& + 4e_2\sin\left(\frac{t}{2}-u\right) - 4e_1\sin\left(\frac{t}{2}+u\right) \\
& \left.\left. + 4e_2\sin\left(\frac{t}{2}+u\right)\right]^2\Bigg\} \Big/ (-8 + 8\cos(t) + 3t\sin(t))^2\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(2\cos(u)\left[(y_{01}-y_{02})\cos\left(\frac{t}{2}\right)\right.\right.\\
&\left.+2(e_1-e_2)\sin(u)\right]\Big/\left[3t\cos\left(\frac{t}{2}\right)-8\sin\left(\frac{t}{2}\right)\right]^2\\
&+\left[4\cos\left(\frac{t}{2}-u\right)-4\cos\left(\frac{t}{2}+u\right)\right.\\
&\left.+3t\sin\left(\frac{t}{2}-u\right)\right]\left[-2(y_{01}-y_{02})\cos(t)\right.\\
&+3(e_1-e_2)t\cos\left(\frac{t}{2}-u\right)+2(y_{01}-y_{02})\\
&\left.-2(e_1-e_2)\sin\left(\frac{t}{2}-u\right)-2(e_1-e_2)\sin\left(\frac{t}{2}+u\right)\right]\\
&\left.\Big/(-8+8\cos(t)+3t\sin(t))^2\right)\Big/\left\{\left[(y_{01}-y_{02})\cos\left(\frac{t}{2}\right)\right.\right.\\
&\left.+2(e_1-e_2)\sin(u)\right]^2\Big/\left[3t\cos\left(\frac{t}{2}\right)-8\sin\left(\frac{t}{2}\right)\right]^2\\
&+\left[-2y_{01}+2y_{02}+2(y_{01}-y_{02})\cos(t)\right.\\
&+3(-e_1+e_2)t\cos\left(\frac{t}{2}-u\right)+4e_1\sin\left(\frac{t}{2}-u\right)\\
&\left.-4e_2\sin\left(\frac{t}{2}-u\right)+4e_1\sin\left(\frac{t}{2}+u\right)-4e_2\sin\left(\frac{t}{2}+u\right)\right]^2\\
&\left.\Big/(-8+8\cos(t)+3t\sin(t))^2\right\}^{\frac{1}{2}} \qquad \text{(A6)}
\end{aligned}
$$

Thus, the condition $t = v$ renders Eq. (A4) or Eq. (A5) equivalent, leaving two equations, that is, for example, Eq. (A4) and

$$\frac{\partial|\Delta\mathbf{v}|}{\partial t} = 0 \qquad \text{(A7)}$$

The conclusion to be drawn is therefore that the total $|\Delta v|$ has a minimum for $t = v$, or when the time of flight is equal to $\phi_2 - \phi_1$, where of course the specific value of t must now be found together with the corresponding value of u from Eqs. (A4) and (A7).

Acknowledgment

The author is grateful for the comments and suggestions offered by the reviewers of this paper.

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